

MONODROMY OF CODIMENSION-ONE SUB-FAMILIES OF UNIVERSAL CURVES

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ABSTRACT. Suppose that $g \geq 3$, that $n \geq 0$ and that $\ell \geq 1$. The main result is that if E is a smooth variety that dominates a codimension 1 subvariety D of $\mathcal{M}_{g,n}[\ell]$, the moduli space of n -pointed, genus g , smooth, projective curves with a level ℓ structure, then the closure of the image of the monodromy representation $\pi_1(E, e_o) \rightarrow \mathrm{Sp}_g(\widehat{\mathbb{Z}})$ has finite index in $\mathrm{Sp}_g(\widehat{\mathbb{Z}})$. A similar result is proved for codimension 1 families of principally polarized abelian varieties.

1. INTRODUCTION

Suppose that g, ℓ, n are integers satisfying $g \geq 2$, $\ell \geq 1$ and $n \geq 0$. Denote the moduli space of smooth complex projective curves of genus g with a level $\ell \geq 1$ structure by $\mathcal{M}_g[\ell]$ and the n th power of the universal curve over it by $\mathcal{C}_g^n[\ell]$. The moduli space of smooth n -pointed smooth projective curves of genus g with a level $\ell \geq 1$ structure $\mathcal{M}_{g,n}[\ell]$ is a Zariski open subset of $\mathcal{C}_g^n[\ell]$. These will all be regarded as orbifolds. There is a natural monodromy representation

$$\rho : \pi_1(\mathcal{C}_g^n[\ell], x_o) \rightarrow \mathrm{Sp}_g(\mathbb{Z})$$

whose image is the level ℓ -congruence subgroup of $\mathrm{Sp}_g(\mathbb{Z})$.

The profinite completion of a discrete group Γ will be denoted by Γ^\wedge . Denote the profinite completion of the integers by $\widehat{\mathbb{Z}}$. A homomorphism $\Gamma \rightarrow \mathrm{GL}_N(\mathbb{Z})$ induces a homomorphism $\Gamma^\wedge \rightarrow \mathrm{GL}_N(\widehat{\mathbb{Z}})$.

Theorem 1. *Suppose that $g = 3$ or $g \geq 5$. If $E \rightarrow D$ is a dominant morphism from a smooth variety to an irreducible divisor D in $\mathcal{C}_g^n[\ell]$, then the image of the monodromy representation*

$$\pi_1(E, e_o)^\wedge \rightarrow \pi_1(D, d_o)^\wedge \rightarrow \mathrm{Sp}_g(\widehat{\mathbb{Z}})$$

has finite index in $\mathrm{Sp}_g(\widehat{\mathbb{Z}})$.

The statement is false when $g = 2$ as will be explained in Section 5. We will prove a stronger version of this theorem (Theorem 5.1), in which $\mathcal{C}_g^n[\ell]$ is replaced by a “suitably generic linear section” of dimension ≥ 3 of it. We also prove similar result for abelian varieties (Theorem 5.2).

Each rational representation V of Sp_g determines a local system \mathbb{V} over $\mathcal{C}_g^n[\ell]$. The theorem implies that when V does not contain the trivial representation, the low dimensional cohomology of $\mathcal{C}_g^n[\ell]$ with coefficients in \mathbb{V} changes very little when $\mathcal{C}_g^n[\ell]$ is replaced by a Zariski open subset of $\mathcal{C}_g^n[\ell]$ or by its generic point.

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Corollary 2. *Suppose that $g = 3$ or $g \geq 5$, $n \geq 0$ and $\ell \geq 1$. If U is a Zariski open subset of $\mathcal{C}_g^n[\ell]$, then for all non-trivial, irreducible representations V of $\mathrm{Sp}(H)$, the map*

$$H^j(\mathcal{C}_g^n[\ell], \mathbb{V}) \rightarrow H^j(U, \mathbb{V})$$

induced by the inclusion $U \hookrightarrow \mathcal{C}_g^n[\ell]$ is an isomorphism when $j = 0, 1$ and an injection when $j = 2$.

The groups $H^j(\mathcal{C}_g^n[\ell], V)$ are known for all V when $j = 1$ and $g \geq 3$; a canonical subspace of it is known [9] when $j = 2$ when $g \geq 6$. The resulting computation of the Galois cohomology groups $H^j(G_{k(\mathcal{C}_g^n[\ell])}, V \otimes \mathbb{Q}_p)$ of the absolute Galois group of the function field $k(\mathcal{C}_g^n[\ell])$, when k is a number field, is an essential ingredient of the author's investigation [12] of rational points of the universal curve over the function field of $\mathcal{M}_{g,n/k}[\ell]$.

The proof of Theorem 1 easily reduces to the case $n = 0$. Putman [18] has proved that the Picard group of $\mathcal{M}_g[\ell]$ has rank 1 when $g \geq 5$. A standard argument using the fact that $\mathcal{M}_g[\ell]$ is quasi-projective then implies that every irreducible divisor D in $\mathcal{M}_g[\ell]$ is ample. A Lefschetz type theorem due to Goresky and MacPherson [8] implies that when D is generic, $\pi_1(D, d_o) \rightarrow \pi_1(\mathcal{M}_g[\ell], x_o)$ is an isomorphism. In this case, the result is immediate. The principal difficulty occurs when D has self-intersections. In this case the image of the homomorphism

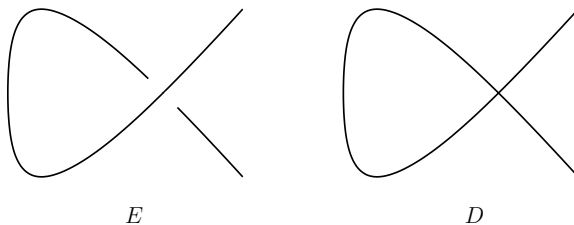


FIGURE 1. Normalization

$\pi_1(E, e_o) \rightarrow \pi_1(D, d_o)$ induced by its normalization $E \rightarrow D$ may have infinite index in $\pi_1(D, d_o)$. This issue is addressed by a technical result, Theorem 2.1, from which Theorem 1 follows directly. It is a hybrid of a “non-abelian strictness theorem” and a Lefschetz type theorem.

The term *non-abelian strictness theorem* needs further explanation. Input for one such type of theorem is a diagram

$$(1) \quad Z \xrightarrow{f} Y \xrightarrow{j} X$$

of morphisms of varieties, where X and Z are smooth, and where f is dominant. Deligne proved several prototypical strictness theorems for cohomology in [7]. In the situation (1), a standard strictness argument, given in Section 9, implies that the image of $H_1(Z) \rightarrow H_1(X)$ has finite index in the image of $H_1(Y) \rightarrow H_1(X)$, even though the image of $H_1(Z)$ may have infinite index in $H_1(Y)$. We will refer to this as an “abelian strictness theorem” as the invariant H_1 is the abelianization of the fundamental group. The most optimistic formulation of a non-abelian strictness assertion would be that the image of $\pi_1(Z, z_o)$ in $\pi_1(X, x_o)$ has finite index in the image of $\pi_1(Y, y_o)$ in $\pi_1(X, x_o)$. A weaker assertion would be that

this holds for profinite fundamental groups. A less optimistic statement would be that, for all reductive linear representations $\pi_1(X, x_o) \rightarrow \mathrm{GL}_N(F)$, the image of $\rho_Z : \pi_1(Z, z_o) \rightarrow \mathrm{GL}_N(F)$ has finite index in the image of $\rho_Y : \pi_1(Y, z_o) \rightarrow \mathrm{GL}_N(F)$, or even that the Zariski closure of the image of ρ_Z has finite index in the Zariski closure of the image of ρ_Y . All four assertions are false, as is shown by the example in Section 9. This, I hope, goes part way towards justifying the technical assumptions of Theorem 2.1.

Conventions: An orbifold is a stack in the category of topological spaces in which the automorphism group of each point is finite. All the orbifolds considered in this paper are quotients of a contractible complex manifold by a virtually free action of a finitely generated discrete group. Equivalently, they are all quotients of a complex manifold by a finite group. A detailed exposition of such orbifolds can be found in [11].

Unless stated to the contrary, all varieties are defined over \mathbb{C} . They will be regarded as topological spaces (or orbifolds) in the complex topology. A divisor in a variety X is a closed subvariety of pure codimension 1.

Implicit in the statement that a map $f : X \rightarrow Y$ induces a homomorphism $\pi_1(X, x_o) \rightarrow \pi_1(Y, y_o)$ is the assertion that $y_o = f(x_o)$.

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I am especially grateful to Madhav Nori who alerted me to a gap in the proof of Lemma 7.5 and contributed its current formulation and proof. He also suggested the approach to proving Theorem 1 in genus 4, pointing out the relevance of his remarkable result [17, Thm. 5.4].

2. SETUP AND STATEMENT OF THE STRICTNESS THEOREM

2.1. Setup. The main objects in the Strictness Theorem are:

- (1) a smooth, connected quasi-projective variety M ,
- (2) a projective completion \overline{M} of M and an open subvariety M^c that contains M ,
- (3) a local system \mathbb{V} over M^c whose fibers are torsion free \mathbb{Z} -modules.

We will suppose that:

- (4) each component of $\overline{M} - M$ has codimension ≥ 2 in \overline{M} and each component of $\overline{M} - M^c$ has codimension ≥ 3 in \overline{M} ,
- (5) M^c is unibranch (i.e., locally analytically irreducible) at the generic point of each codimension 2 stratum of $M^c - M$;

Choose a basepoint x_o of M . Denote the fiber of \mathbb{V} over $f(x_o)$ by V_o . Set $\Gamma_o = \text{Aut } V_o$. For each $m \geq 1$, set

$$\Gamma_o[m] = \{\phi \in \text{Aut } V_o : \phi \equiv \text{id mod } m\}.$$

The inverse image of $\Gamma_o[m]$ under the monodromy representation $\rho : \pi_1(M, x_o) \rightarrow \Gamma_o$ corresponds to a connected, unramified, Galois covering $p_m : M_m \rightarrow M$.

(6) $\text{Pic } M_m$ is finitely generated of rank 1 for all $m \geq 1$.

Note that if $\dim M = 1$, then $M = \mathbb{P}^1$.

2.2. An example. The motivating example, to be explained in detail in Section 4, is:

- M is $\mathcal{M}_g[\ell]$, the moduli space of smooth projective curves of genus $g \geq 5$ with a level $\ell \geq 3$ structure;
- \overline{M} is its Satake compactification and M^c is the open subset whose points correspond to genus g curves of compact type;
- \mathbb{V} is the local system over M^c whose fiber over the moduli point of the curve C is the first cohomology $H^1(\text{Jac } C; \mathbb{Z})$ of its jacobian.

In this example $M_m = \mathcal{M}_g[m']$ where $m' = \text{lcm}(m, \ell)$.

Another example is proved by taking $M = M^c$ to be the moduli space $\mathcal{A}_g[\ell]$ of principally polarized abelian varieties with a level $\ell \geq 3$ structure, where $g \geq 3$. Details will be given in Section 4.

2.3. Statement of the Strictness Theorem.

Theorem 2.1. *Suppose that D is an irreducible divisor in M and that $h : E \rightarrow D$ is a dominant morphism. If the conditions in Section 2.1 are satisfied, then the image of $\pi_1(E, e_o)^\wedge \rightarrow \Gamma_o^\wedge$ has finite index in the image of $\pi_1(M, x_o)^\wedge \rightarrow \Gamma_o^\wedge$.*

It is not clear that all of these hypotheses are necessary. Inspired by the abelian case (Proposition 9.1), one might suspect that the image of $\pi_1(E, e_o)^\wedge \rightarrow \pi_1(M, x_o)^\wedge$ has finite index in the image of $\pi_1(D, x_o)^\wedge \rightarrow \pi_1(M, x_o)^\wedge$. Unfortunately, this is not the case as we show in Section 8, where we present examples that suggest that most of the hypotheses above are necessary.

3. TOPOLOGICAL AND GEOMETRIC PRELIMINARIES

3.1. Topological preliminaries. One of the main tools used to prove the Strictness Theorem is a Lefschetz type theorem proved by Goresky and MacPherson in their book [8] on stratified Morse theory. For the convenience of the reader, we state the special case of this theorem that we will be using repeatedly.

Fix a real analytic riemannian metric on \mathbb{P}^N . For a subset A of \mathbb{P}^N and a real number $\delta > 0$, set

$$A_\delta := \{x \in \mathbb{P}^N : \text{dist}(x, A) < \delta\}.$$

Theorem 3.1 (Goresky and MacPherson [8, p. 150]). *Suppose that X is an n -dimensional, connected complex algebraic manifold and that $f : X \rightarrow \mathbb{P}^N$ is a morphism, all of whose fibers are finite. If L is a codimension c linear subspace of \mathbb{P}^N , then for all sufficiently small $\delta > 0$, the homomorphism*

$$(2) \quad \pi_j(f^{-1}(L_\delta), x) \rightarrow \pi_j(X, x)$$

induced by the inclusion is an isomorphism when $j < n(c)$ and surjective when $j = n(c)$, where $n(c) = \max\{n - c, -1\}$. If L is generic or if $f^{-1}(L)$ is proper, then L_δ may be replaced by L in (2).

The final statement is slightly stronger than that made on page 151 of [8]. The stronger statement follows from a result of Durfee [6] which implies that when $f^{-1}(L)$ is compact, $f^{-1}(L_\delta)$ is a regular neighbourhood of $f^{-1}(L)$, so that the inclusion $f^{-1}(L) \hookrightarrow f^{-1}(L_\delta)$ is a homotopy equivalence.

The theorem implies that generic linear sections of X of dimension ≥ 3 have the same Picard group.

Corollary 3.2. *Assume the notation of the theorem above. If L is a generic linear subspace of \mathbb{P}^N of codimension $\leq \dim X - 3$, then the inclusion $f^{-1}(L) \rightarrow X$ induces an isomorphism*

$$\text{Pic } X \rightarrow \text{Pic } f^{-1}(L).$$

Proof. Since X and $f^{-1}(L)$ are smooth, the assertion follows from Theorem 3.1 and the fact that for a complex algebraic manifold Y the sequence

$$(3) \quad 0 \rightarrow \text{Ext}_{\text{Hodge}}^1(\mathbb{Z}, H^1(Y, \mathbb{Z}(1))) \rightarrow \text{Pic } Y \rightarrow \text{Hom}_{\text{Hodge}}(\mathbb{Z}(0), H^2(Y, \mathbb{Z}(1))) \rightarrow 0$$

is exact. \square

A direct consequence is that the hypotheses of Theorem 2.1 behave well when taking generic linear sections of dimension ≥ 3 .

Corollary 3.3. *Suppose that the members of the diagram*

$$M \hookrightarrow M^c \hookrightarrow \overline{M}$$

and the local system \mathbb{V} over M^c satisfy the conditions in Section 2.1. If $\overline{M} \rightarrow \mathbb{P}^N$ is a projective imbedding and L is a generic linear subspace of \mathbb{P}^N such that $\dim(M \cap L) \geq 3$, then the members of the diagram

$$M \cap L \hookrightarrow M^c \cap L \hookrightarrow \overline{M} \cap L$$

and the restriction of \mathbb{V} to $M^c \cap L$ also satisfy the conditions in Section 2.1.

The following fact will be used in the proof the the Strictness Theorem.

Corollary 3.4. *If the members of*

$$M \hookrightarrow M^c \hookrightarrow \overline{M}$$

satisfy the conditions in Section 2.1, then M contains a smooth complete curve and every irreducible divisor in M is ample.

Proof. Fix a projective imbedding $f : \overline{M} \hookrightarrow \mathbb{P}^N$. Since each component of $\overline{M} - M$ has codimension ≥ 2 , the intersection $M \cap L$ of a generic linear subspace L of \mathbb{P}^N of codimension $\dim M - 1$ is a complete curve in M . If D is an irreducible divisor in M , then the image of its closure $f(\overline{D})$ will have degree $L \cdot \overline{f}_* \overline{D}$ in \mathbb{P}^N . This degree is positive. Since $\text{Pic } M$ is finitely generated of rank 1, D is ample in M . \square

3.2. Dominant morphisms. This section contains a few elementary results that are needed in the proof of the strictness theorem.

The link of every stratum of a complex algebraic variety is connected if and only if X is unibranch. From this it follows that:

Lemma 3.5. *If U is a Zariski open subset of an irreducible, unibranch variety X , then the induced homomorphism $\pi_1(U, x_o) \rightarrow \pi_1(X, x_o)$ is surjective.* \square

The following is standard. Cf. [5, p. 363].

Lemma 3.6. *Every normal variety is unibranch.* \square

Suppose D is an irreducible algebraic variety. Denote its normalization by $\tilde{D} \rightarrow D$.

Proposition 3.7. *If $f : E \rightarrow D$ is a dominant morphism from a variety to D , then the image of $\pi_1(E, e_o) \rightarrow \pi_1(D, x_o)$ contains a finite index subgroup of the image of $\pi_1(\tilde{D}, x_o) \rightarrow \pi_1(D, x_o)$. Here $e_o \in E$ is chosen so that $x_o = f(e_o)$ is a smooth point of D .*

Proof. Let $\tilde{E} \rightarrow E$ be the normalization of E . The normalization of D in the function field of \tilde{E} is its normalization \tilde{D} . Since the diagram

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{D} & \longrightarrow & D \end{array}$$

commutes, it suffices to show that the image of the homomorphism $\pi_1(\tilde{E}, e_o) \rightarrow \pi_1(\tilde{D}, x_o)$ has finite index in $\pi_1(\tilde{D}, x_o)$. There is a smooth Zariski open subset U of \tilde{D} and a smooth Zariski open subset V of $\tilde{f}^{-1}(U)$ such that the restriction of \tilde{f} to V is a locally trivial fiber bundle. Since the number of connected components of each fiber is finite, the image of $\pi_1(V, e_o) \rightarrow \pi_1(U, x_o)$ has finite index in $\pi_1(U, x_o)$. The result now follows as the homomorphisms $\pi_1(V, e_o) \rightarrow \pi_1(\tilde{E}, e_o)$ and $\pi_1(U, x_o) \rightarrow \pi_1(\tilde{D})$ are surjective by Lemma 3.6 and Lemma 3.5. \square

Specializing to the case where D is smooth, we obtain the following special case.

Corollary 3.8. *Suppose that Y is a smooth variety and that $f : X \rightarrow Y$ is a dominant morphism. If E is an irreducible divisor in X and if the restriction of f to E is dominant, then the image of $\pi_1(E, e_o) \rightarrow \pi_1(Y, y_o)$ has finite index in $\pi_1(Y, y_o)$.*

3.3. Lean morphisms. The Strictness Theorem applies to more general situations than those described in Section 2. In order to describe them, we will need the following definition. Recall that, by a divisor in an irreducible variety X , we mean a closed subvariety of pure codimension 1.

Definition 3.9. A dominant morphism $F : X \rightarrow Y$ is *lean* if the codimension of the inverse image $F^{-1}(Z)$ of an irreducible closed subvariety Z of Y is a divisor in X implies that Z is a divisor in Y .

Every dominant morphism $F : X \rightarrow Y$ all of whose closed fibers have the same dimension, is lean. In particular, flat morphisms are lean.

Lemma 3.10. *Suppose that the strictness theorem (Theorem 2.1) holds for the local system \mathbb{V} over M . If $h : X \rightarrow M$ is a lean morphism, then the strictness theorem holds for the pullback of \mathbb{V} to X . That is, if D is an irreducible divisor in X , and if $E \rightarrow D$ is a dominant morphism from a smooth variety to D , then the image of $\pi_1(E, e_o)^\wedge \rightarrow \Gamma_o^\wedge$ has finite index in the image of $\pi_1(M, x_o)^\wedge \rightarrow \Gamma_o^\wedge$.*

Proof. Since h is lean, the image of D in M has codimension ≤ 1 . When $D \rightarrow M$ is dominant, so is $E \rightarrow M$. Corollary 3.8 implies that the image of $\pi_1(E, e_o)$ in $\pi_1(M, x_o)$ has finite index in $\pi_1(M, x_o)$, which proves the result in this case. If the closure F of the image of D in M is a divisor, then $E \rightarrow F$ is dominant so that the hypothesis that the strictness theorem holds for M implies that the image of

$$\pi_1(E, e_o)^\wedge \rightarrow \pi_1(M, x_o)^\wedge \rightarrow \Gamma_o^\wedge$$

has finite index in the image of $\pi_1(M, x_o)^\wedge \rightarrow \Gamma_o^\wedge$. \square

The proof of the following proposition and the lemma were contributed by the referee in response to a question in an earlier version of this paper. They guarantee that generic hypersurface sections of a flat family of varieties are lean.

Lemma 3.11. *If Y is an algebraic subvariety of \mathbb{P}^N of positive dimension d , then the locus in $\mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}(e)))$ of hypersurfaces of degree e that contain Y has codimension at least $e^d/d!$.*

Proof. Replace Y by Y^{red} if necessary. Let y be a smooth point of Y . Since $\dim Y = d$,

$$\dim H^0(\text{Spec } \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^{e+1}, \mathcal{O}(e)) = \binom{d+e}{e} \geq e^d/d!$$

The result follows as the composite of the maps

$$H^0(\mathbb{P}^N, \mathcal{O}(e)) \xrightarrow{\simeq} H^0(\text{Spec } \mathcal{O}_{\mathbb{P}^N,y}/\mathfrak{m}_{\mathbb{P}^N,y}^{e+1}, \mathcal{O}(e)) \longrightarrow H^0(\text{Spec } \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^{e+1}, \mathcal{O}(e))$$

is surjective. \square

Proposition 3.12. *Suppose that X and T are quasi-projective varieties. Fix an imbedding $X \rightarrow \mathbb{P}^N$. If $f : X \rightarrow T$ is a dominant morphism, all of whose fibers have the same dimension $d > 0$, then there exists $e > 0$ such that for a generic hypersurface Y of \mathbb{P}^N of degree e , the restriction $f_S : S \rightarrow T$ of f to $S := Y \cap X$ is a dominant morphism each of whose fibers has dimension $d - 1$. In particular, f_S is lean.*

Proof. For $e > 0$, let Z be the subvariety of $\mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}(e))) \times T$ consisting of pairs (Y, t) where the degree d hypersurface Y contains the fiber of f over $t \in T$. The lemma above implies that Z has codimension $\geq e^d/d!$. Denote by W the Zariski closure of the image of Z under the projection of $\mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}(e))) \times T$ onto its first factor. This has codimension $\geq e^d/d! - \dim T$. If we choose e such that $e^d/d! > \dim T$, then W will be a proper subvariety of $\mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}(e)))$, which implies the result. \square

By taking iterated generic hypersurface sections, we conclude:

Corollary 3.13. *Suppose that X and T are quasi-projective varieties. Fix an imbedding $X \rightarrow \mathbb{P}^N$. If $f : X \rightarrow T$ is a dominant morphism, all of whose fibers have the same dimension $d > 0$, then for a section of X by a generic complete*

intersection of codimension $c \leq \text{codim} T$ of multi-degree (e_1, \dots, e_c) , where $0 \ll e_1 \ll e_2 \ll \dots \ll e_c$, the restriction $f_S : S \rightarrow T$ of f to $S := L \cap X$ is a dominant morphism each of whose fibers has dimension $d - c$. In particular, f_S is lean.

4. MODULI SPACES AND THEIR BAILY-BOREL-SATAKE COMPACTIFICATIONS

Suppose that $g \geq 1$ and that $\ell \geq 1$. Following work of Satake [19], Baily and Borel [2] constructed a minimal compactification X^* of each locally symmetric variety X . We will call X^* the *Satake compactification* of X . It is a normal complex projective algebraic variety. The Satake compactification $\mathcal{A}_g^*[\ell]$ of $\mathcal{A}_g[\ell]$ has the property that its boundary $\mathcal{A}_g^*[\ell] - \mathcal{A}_g[\ell]$ has complex codimension g .

The only imbeddings of $\mathcal{A}_g[\ell]$ into projective space that we will consider are ones that extend to imbeddings of $\mathcal{A}_g^*[\ell]$. All such imbeddings are given by automorphic forms of a sufficiently high weight. By a linear section of $\mathcal{A}_g[\ell]$, we mean a linear section with respect to such an imbedding.

Proposition 4.1. *If $g \geq 2$ and $\ell \geq 3$, then $\mathcal{M}_g[\ell]$ has a projective completion $\mathcal{M}_g^*[\ell]$ such that the period mapping $f : \mathcal{M}_g[\ell] \rightarrow \mathcal{A}_g[\ell]$ extends to a finite morphism*

$$\bar{f} : \mathcal{M}_g^*[\ell] \rightarrow \mathcal{A}_g^*[\ell].$$

When $g \geq 3$, the subvariety $\bar{f}^{-1}(\mathcal{A}_g^[\ell] - \mathcal{A}_g[\ell])$ of $\mathcal{M}_g^*[\ell]$ has codimension 3 and $\mathcal{M}_g^*[\ell]$ is unibranch at the generic point of each codimension 2 stratum of $\mathcal{M}_g^*[\ell] - \mathcal{M}_g[\ell]$.*

Proof. Define $\mathcal{M}_g^*[\ell]$ to be the normalization of $\mathcal{A}_g^*[\ell]$ in the function field of $\mathcal{M}_g[\ell]$. Since $\mathcal{M}_g[\ell]$ is smooth (and thus normal) when $\ell \geq 3$, $\mathcal{M}_g[\ell]$ is a subvariety of $\mathcal{M}_g^*[\ell]$. The projection $\bar{f} : \mathcal{M}_g^*[\ell] \rightarrow \mathcal{A}_g^*[\ell]$ is finite. The generic point of a boundary component of $\mathcal{A}_g^*[\ell]$ is a principally polarized abelian variety of dimension $g - 1$ with a level ℓ structure. The generic point of the image of the boundary $\mathcal{M}_g^*[\ell] - \mathcal{M}_g[\ell]$ is the jacobian of a smooth projective curve of genus $g - 1$, which has dimension $3g - 6$. Since \bar{f} is finite, this implies that $\bar{f}^{-1}(\mathcal{A}_g^*[\ell] - \mathcal{A}_g[\ell])$ has codimension 3 in $\mathcal{M}_g^*[\ell]$.

The moduli space \mathcal{M}_g^c of genus g complex projective curves of compact type is the complement of the divisor Δ_0 in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of \mathcal{M}_g . It will be regarded as an orbifold. The period mapping extends to a proper mapping $\mathcal{M}_g^c \rightarrow \mathcal{A}_g$. Since

$$\pi_1(\mathcal{M}_g, x_o) \rightarrow \pi_1(\mathcal{M}_g^c, x_o) \rightarrow \pi_1(\mathcal{A}_g, a_o) \cong \text{Sp}_g(\mathbb{Z})$$

is surjective, so is $\pi_1(\mathcal{M}_g^c, x_o) \rightarrow \pi_1(\mathcal{A}_g, a_o) \cong \text{Sp}_g(\mathbb{Z})$. Denote the Galois covering of \mathcal{M}_g^c corresponding to the kernel of the homomorphism $\pi_1(\mathcal{M}_g^c, x_o) \rightarrow \text{Sp}_g(\mathbb{Z}/\ell\mathbb{Z})$ by $\mathcal{M}_g^c[\ell]$. This is a smooth orbifold that contains $\mathcal{M}_g[\ell]$ as a Zariski open subset. Since $\mathcal{M}_g^c[\ell]$ is normal, the period mapping

$$\mathcal{M}_g^c[\ell] \rightarrow \mathcal{A}_g[\ell] \hookrightarrow \mathcal{A}_g^*[\ell]$$

factors through $\mathcal{M}_g^*[\ell] \rightarrow \mathcal{A}_g^*[\ell]$. The fiber of $\mathcal{M}_g^c[\ell] \rightarrow \mathcal{M}_g[\ell]$ over the generic point of a codimension 2 stratum of $\mathcal{M}_g^*[\ell] - \mathcal{M}_g[\ell]$ is a smooth projective curve of genus $g - 1$. The link of the image of such a genus $g - 1$ curve C in $\mathcal{M}_g^*[\ell]$ is the link of C in $\mathcal{M}_g^c[\ell]$, which is the unit tangent bundle of C and is therefore connected. It follows that $\mathcal{M}_g^c[\ell]$ is unibranch at the generic point of each codimension 2 component of $\mathcal{M}_g^*[\ell] - \mathcal{M}_g[\ell]$. \square

5. MONODROMY THEOREMS

There are two significant classes of examples where the hypotheses of Theorem 2.1 are satisfied and in which we can prove Theorem 1 and generalizations. These are constructed from moduli spaces of curves, and from moduli spaces of abelian varieties. In both cases the local system \mathbb{V} over M is pulled back from the local system over \mathcal{A}_g whose fiber over the moduli point of the abelian variety A is $H^1(A; \mathbb{Z})$. In both cases the image of the monodromy homomorphism is a finite index subgroup of

$$\text{Aut}(H^1(A, \mathbb{Z}), \theta)$$

where $\theta : H^1(A, \mathbb{Z})^{\otimes 2} \rightarrow \mathbb{Z}$ denotes the polarization. The choice of a symplectic basis of $H^1(A; \mathbb{Z})$ gives an isomorphism of this group with $\text{Sp}_g(\mathbb{Z})$, the group of $2g \times 2g$ integral symplectic matrices.

5.1. Curves. Suppose that $g \geq 5$ and that $\ell \geq 3$. Then $\mathcal{M}_g[\ell]$ is a smooth quasi-projective variety. Fix an imbedding $\mathcal{M}_g^*[\ell] \hookrightarrow \mathbb{P}^N$ of the Satake compactification of $\mathcal{M}_g[\ell]$. Suppose that L is a linear subspace of \mathbb{P}^N that is generic with respect to the inclusion $f : \mathcal{M}_g^*[\ell] \hookrightarrow \mathbb{P}^N$, and that satisfies $\dim(L \cap \mathcal{M}_g[\ell]) \geq 3$. Set $M = f^{-1}(L)$. This is a smooth subvariety of $\mathcal{M}_g[\ell]$ of dimension ≥ 3 .

Suppose that $n \geq 0$. Denote the restriction of the universal curve over $\mathcal{M}_g[\ell]$ to M by \mathcal{C}_M and its n th power by \mathcal{C}_M^n . This is a quasi-projective variety. Fix an imbedding $\mathcal{C}_M^n \hookrightarrow \mathbb{P}^r$.

Theorem 5.1. *If X is a generic section of \mathcal{C}_M^N in \mathbb{P}^r by a complete intersection of codimension $c \leq \text{codim } M$ and multi-degree (e_1, \dots, e_c) , where $0 \ll e_1 \ll e_2 \ll \dots \ll e_c$, then*

- (1) X is smooth,
- (2) if D is an irreducible divisor in X and $E \rightarrow D$ is a dominant morphism from a smooth variety to D , then the image of $\pi_1(E, e_o)^\wedge \rightarrow \text{Sp}_g(\widehat{\mathbb{Z}})$ is a finite index subgroup of $\text{Sp}_g(\widehat{\mathbb{Z}})$.

Theorem 1 follows from the case $M = \mathcal{M}_g[\ell]$.

Proof. Proposition 4.1 and Putman's computation of the Picard groups of the $\mathcal{M}_g[m]$ implies that \mathbb{V} and the members of

$$\mathcal{M}_g[\ell] \hookrightarrow \mathcal{M}_g[\ell]^c \hookrightarrow \mathcal{M}_g^*[\ell]$$

satisfy the hypotheses of Theorem 2.1. Corollary 3.3 implies that for generic linear subspaces L of \mathbb{P}^N such that $\dim(L \cap \mathcal{M}_g[\ell]) \geq 3$, the members of

$$\mathcal{M}_g[\ell] \cap L \hookrightarrow \mathcal{M}_g[\ell]^c \cap L \hookrightarrow \mathcal{M}_g^*[\ell] \cap L$$

also satisfy the hypotheses of Theorem 2.1. We will assume that L is such a subspace. Set $M = \mathcal{M}_g[\ell]$, $M^c = \mathcal{M}_g^c[\ell]$ and $\overline{M} = \mathcal{M}_g^*[\ell]$. Theorem 3.1 implies that the inclusion $M \hookrightarrow \mathcal{M}_g[\ell]$ induces an isomorphism on fundamental groups, so that the image of $\pi_1(M, x_o) \rightarrow \text{Sp}_g(\mathbb{Z})$ is $\text{Sp}_g(\mathbb{Z})[\ell]$.

Corollary 3.13 implies that $X \rightarrow M$ is lean. Lemma 3.10 implies that the image of $\pi_1(E, e_o)^\wedge$ in $\text{Sp}_g(\widehat{\mathbb{Z}})$ is a finite index subgroup. \square

5.2. Abelian varieties. Suppose that $g \geq 3$ and that $\ell \geq 3$. Then $\mathcal{A}_g[\ell]$ is a smooth smooth quasi-projective variety. Fix any imbedding

$$\mathcal{A}_g^*[\ell] \hookrightarrow \mathbb{P}^N$$

of the Satake compactification of $\mathcal{A}_g[\ell]$ given by automorphic forms. Suppose that L is a linear subspace of \mathbb{P}^N that is generic with respect to the imbedding $\mathcal{A}_g^*[\ell] \hookrightarrow \mathbb{P}^N$ and that satisfies $\dim(L \cap \mathcal{A}_g^*[\ell]) \geq 3$. Set $M = L \cap \mathcal{A}_g[\ell]$. This is a smooth subvariety of $\mathcal{A}_g[\ell]$ of dimension ≥ 3 .

Suppose that $n \geq 0$. Denote the restriction of the universal abelian variety over $\mathcal{A}_g[m]$ to M by \mathcal{X}_M and its n th power by \mathcal{X}_M^n . This is a quasi-projective variety. Fix an imbedding $\mathcal{X}_M^n \hookrightarrow \mathbb{P}^r$.

Theorem 5.2. *If X is a generic section of \mathcal{X}_M^N in \mathbb{P}^r by a complete intersection of codimension $c \leq \text{codim } M$ and multi-degree (e_1, \dots, e_c) , where $0 \ll e_1 \ll e_2 \ll \dots \ll e_c$, then*

- (1) *X is smooth,*
- (2) *if D is an irreducible divisor in X and $E \rightarrow D$ is a dominant morphism from a smooth variety to D , then the image of $\pi_1(E, e_o)^\wedge \rightarrow \text{Sp}_g(\widehat{\mathbb{Z}})$ is a finite index subgroup of $\text{Sp}_g(\widehat{\mathbb{Z}})$.*

The assumptions are satisfied when $X = \mathcal{X}_M^N$. The case where $M = \mathcal{A}_g[\ell]$ is an analogue of Theorem 1 for principally polarized abelian varieties.

Proof. Borel's computation of the stable cohomology of arithmetic group implies that for all $g \geq 3$ and $\ell \geq 1$, $H^1(\mathcal{A}_g[\ell], \mathbb{Z})$ vanishes and $H^2(\mathcal{A}_g[\ell], \mathbb{Z})$ is finitely generated of rank 1. The exact sequence (3) now implies that $\text{Pic } \mathcal{A}_g[m]$ is finitely generated of rank ≤ 1 for all $m \geq \ell$. But since the determinant of the Hodge bundle has non-trivial Chern class, it follows that $\text{Pic } \mathcal{A}_g[m]$ has rank 1 for all $m \geq \ell$. This implies that the members of

$$\mathcal{A}_g[\ell] = \mathcal{A}_g^c[\ell] \hookrightarrow \mathcal{A}_g^*[\ell]$$

satisfy the hypotheses of Theorem 2.1. The rest of the proof is almost identical with that of Theorem 5.1 and is left to the reader. \square

5.3. Proof of Theorem 1. When $g \geq 5$ and $\ell \geq 3$, this follows from Theorem 5.1 by taking $M = \mathcal{M}_g[\ell]$ and $X = \mathcal{C}_g^n[\ell]$. The cases where $\ell = 1, 2$ are both immediate consequences of the case $\ell = 4$.

Suppose that $g = 3$ and that D is an irreducible divisor in $\mathcal{M}_3[\ell]$. Since the period mapping $f : \mathcal{M}_g[\ell] \rightarrow \mathcal{A}_3[\ell]$ is quasi-finite and dominant, the Zariski closure \overline{D} of the image of D in $\mathcal{A}_3[\ell]$ is a divisor in $\mathcal{A}_3[\ell]$. If $E \rightarrow D$ is dominant, then $E \rightarrow \overline{D}$ is also dominant. Theorem 5.2 now implies that when E is smooth, the image of $\pi_1(E, e_o)^\wedge$ in $\text{Sp}_3(\widehat{\mathbb{Z}})$ has finite index.

It remains to prove the genus 4 case. Here we take a different approach suggested by Nori. It suffices to prove the result when $\ell \geq 3$. Let G be the Zariski closure of the image of $\pi_1(E, e_o)$ in the \mathbb{Q} -group

$$\text{Aut}(H_1(C_o, \mathbb{Q}), \theta) \cong \text{Sp}_{4/\mathbb{Q}},$$

where C_o denotes the curve corresponding to the point $x_o \in \mathcal{M}_4$ and θ its polarization. Standard arguments imply that G is of Hodge type. In particular, its Lie algebra \mathfrak{g} is a sub Hodge structure of the Lie algebra $S^2 H_1(C_o)$ of $\text{Aut}(H_1(C_o, \mathbb{Q}), \theta)$.

The associated symmetric space $\mathcal{D} := G(\mathbb{R})/K_G$ is hermitian as its tangent space $\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g} \cong \mathfrak{g}^{-1,1}$ at the base point is a complex subspace of the tangent space of \mathfrak{h}_g , the symmetric space of $\mathrm{Sp}_4(\mathbb{R})$. Set $\Gamma = \mathrm{Sp}_4(\mathbb{Z})[\ell] \cap G(\mathbb{R})$. Then $X := \Gamma \backslash \mathcal{D}$ is a locally symmetric subvariety of $\mathcal{A}_4[\ell]$. The theorem of the fixed part implies that X contains the image of D in $\mathcal{A}_4[\ell]$ from which it follows that $\dim X \geq \dim D = 8$.

Write \mathcal{D} as a product $\prod_{j=1}^N \mathcal{D}_j$ of irreducible, symmetric spaces $\mathcal{D}_j = G_j(\mathbb{R})/K_j$, where G_j is a real Lie group and K_j is a maximal compact subgroup. Each \mathcal{D}_j is hermitian, [13, p. 381]. The classification of hermitian symmetric spaces [13, p. 518] implies that G_j is isogenous to $\mathrm{SU}(p, q)$ with $p + q \leq 5$, to $\mathrm{SO}_o(n, 2)$ with $n \leq 7$, to $\mathrm{Sp}_n(\mathbb{R})$ with $n \leq 4$, or to $\mathrm{SO}^*(2n)$ with $n \leq 4$.

G	$\mathrm{SU}(p, q)$	$\mathrm{SO}_o(n, 2)$	$\mathrm{Sp}_n(\mathbb{R})$	$\mathrm{SO}^*(2n)$
$\dim_{\mathbb{C}} G/K$	pq	n	$n(n+1)/2$	n
$\mathrm{rank}_{\mathbb{C}} G_{\mathbb{C}}$	$p + q - 1$	$1 + \lfloor n/2 \rfloor$	n	$n(n-1)/2$

Let $r_j = \mathrm{rank}_{\mathbb{C}} G_j$, the rank of the complexification of the Lie algebra of G_j , and let $d_j = \dim_{\mathbb{C}} \mathcal{D}_j$. Since $\prod G_j \subseteq \mathrm{Sp}_4$ and since $\prod \mathcal{D}_j \subseteq \mathfrak{h}_4$, we have

$$r_1 + \cdots + r_N \leq \mathrm{rank}_{\mathbb{C}} \mathrm{Sp}_4 = 4 \text{ and } 8 \leq d_1 + \cdots + d_N = \dim \mathcal{D} \leq 10.$$

The only solution is $N = 1$ and $G = \mathrm{Sp}_4(\mathbb{R})$, which forces X to be $\mathcal{A}_4[\ell]$. Nori's result [17, Thm. 5.4] now implies that the closure of the image of $\pi_1(E, e_o)$ in $\mathrm{Sp}_4(\widehat{\mathbb{Z}})$ has finite index.

5.4. Genus 2. As remarked in the introduction, Theorem 1 (and hence Theorem 5.1) does not hold when $g = 2$. Theorem 5.2 also fails when $g = 2$.

Fix $\ell \geq 3$. Denote the Siegel upper half space of rank g by \mathfrak{h}_g :

$$\mathfrak{h}_g = \{Z \in \mathcal{M}_g(\mathbb{C}) : Z = Z^T \text{ and } \mathrm{Im} Z > 0\}.$$

This is the symmetric space of $\mathrm{Sp}_g(\mathbb{R})$. When $g = 1$, it is the usual upper half plane $\{\tau \in \mathbb{C} : \mathrm{Im} \tau > 0\}$. For each $\ell \geq 1$ we have

$$\mathcal{A}_g[\ell] = \mathrm{Sp}_g(\mathbb{Z})[\ell] \backslash \mathfrak{h}_g.$$

View $\mathfrak{h}_1 \times \mathfrak{h}_1$ as a submanifold of \mathfrak{h}_2 via the inclusion

$$(\tau_1, \tau_2) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.$$

The locus in \mathfrak{h}_2 of (framed) principally polarized abelian surfaces that are the product (as a polarized abelian variety) of two elliptic curves is

$$\mathfrak{h}_2^{\mathrm{red}} = \bigcup_{\gamma \in \mathrm{Sp}_2(\mathbb{Z})} \gamma(\mathfrak{h}_1 \times \mathfrak{h}_1).$$

For each $\ell \geq 1$, the locus of reducible abelian surfaces $\mathcal{A}_2^{\mathrm{red}}[\ell]$ is the image in $\mathcal{A}_2[\ell]$ of $\mathfrak{h}_2^{\mathrm{red}}$. When $\ell \geq 3$ the period mapping induces an isomorphism¹

$$\mathcal{M}_2[\ell] \cong \mathcal{A}_2[\ell] - \mathcal{A}_2[\ell]^{\mathrm{red}}.$$

This is well-known — cf. [10, Prop. 6].

Choose $g \in \mathrm{Sp}_2(\mathbb{Q})$ so that $g(\mathfrak{h}_1 \times \mathfrak{h}_1) \not\subseteq \mathfrak{h}_2^{\mathrm{red}}$. This is possible because $\mathrm{Sp}_2(\mathbb{R})$ acts transitively on \mathfrak{h}_2 and because $\mathrm{Sp}_2(\mathbb{Q})$ is dense in $\mathrm{Sp}_2(\mathbb{R})$ in the classical topology. Set

$$\Gamma = (\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \cap g^{-1}\mathrm{Sp}_2(\mathbb{Z})[\ell]g.$$

¹If interpreted in the category of orbifolds, this is true for all $\ell \geq 1$.

This is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Q})^2$. Set $E^c = \Gamma \backslash (\mathfrak{h}_1 \times \mathfrak{h}_1)$ and define $f : E^c \rightarrow \mathcal{A}_2[\ell]$ to be the morphism induced by

$$(\tau_1, \tau_2) \mapsto g \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.$$

When $\ell \geq 3$, Γ is torsion free and E^c is smooth. Set $E = f^{-1}(\mathcal{M}_2[\ell])$. This is a Zariski open subset of E^c . When $\ell \geq 3$, the homomorphism $\pi_1(E, e_o) \rightarrow \pi_1(E^c, e_o)$ is surjective. The Zariski closure of the image of $\pi_1(E, e_o) \rightarrow \pi_1(\mathcal{A}_2[\ell])$ is $g\Gamma g^{-1}$, which has Zariski closure $g\mathrm{SL}_2(\mathbb{Q})^2 g^{-1}$ in $\mathrm{Sp}_2(\mathbb{Q})$.

Let

$$D^c = \text{image of } E^c \text{ in } \mathcal{A}_2[\ell].$$

This is a codimension one algebraic subvariety of $\mathcal{A}_2[\ell]$. Set $D = D^c \cap \mathcal{M}_2[\ell]$. The choice of g guarantees that D is non-empty.

When $\ell \geq 3$, the diagram $E \rightarrow D \hookrightarrow \mathcal{M}_2[\ell]$ is a counter example to Theorem 1 and the diagram $E^c \rightarrow D^c \hookrightarrow \mathcal{A}_2[\ell]$ is a counter example to Theorem 5.2 when $g = 2$ as the image of $\pi_1(E, e_o) \rightarrow \mathrm{Sp}_2(\mathbb{Q})$ is not Zariski dense for.

6. COHOMOLOGICAL APPLICATIONS

Suppose that the members of the diagram

$$M \hookrightarrow M^c \hookrightarrow \overline{M}$$

and the local system \mathbb{V} over M^c satisfy the conditions in Section 2.1. Fix a prime number p . Denote the Zariski closure of the image of the monodromy homomorphism $\pi_1(M, x_o) \rightarrow \mathrm{Aut}(V_o \otimes \mathbb{Q}_p)$ by R . This is a \mathbb{Q}_p -group. Denote by \mathbb{W} the local system of \mathbb{Q}_p vector spaces over M that corresponds to the R -module W .

Theorem 6.1. *Suppose that R is a connected reductive group and that $X \rightarrow M$ is a lean morphism. If U is Zariski open subset of X , then for all non-trivial, irreducible R -modules W , the restriction mapping*

$$H^j(X, \mathbb{V}) \rightarrow H^j(U, \mathbb{V})$$

is an isomorphism when $j = 0, 1$ and an injection when $j = 2$.

Proof. Write

$$X - U = Z \cup \bigcup_{\alpha} D_{\alpha}$$

where Z is a closed subvariety of X , each of whose components has codimension ≥ 2 , and where each D_{α} is an irreducible divisor in X . Since W is a non-trivial irreducible representation of R , Lemma 3.10 implies that $H^0(D_{\alpha}^o, \mathbb{W})$ vanishes for all α , where D_{α}^o denotes the smooth locus of D_{α} . The result now follows from the exactness of the Gysin sequence

$$0 \rightarrow H^1(X, \mathbb{W}) \rightarrow H^1(U, \mathbb{W}) \rightarrow \bigoplus_{\alpha} H^0(D_{\alpha}^o, \pi_{\alpha}^* \mathbb{W}) \rightarrow H^2(X, \mathbb{W}) \rightarrow H^2(U, \mathbb{W}).$$

□

Since all irreducible representations of $\mathrm{Sp}(H)$ are defined over \mathbb{Q} , we conclude:

Corollary 6.2. *Suppose that $g = 3$ or $g \geq 5$, $n \geq 0$ and $m \geq 1$. If U is a Zariski open subset of $\mathcal{C}_g^n[m]$, then for all non-trivial, irreducible representations W of $\mathrm{Sp}(H_{\mathbb{Q}})$, the map*

$$H^j(\mathcal{C}_g^n[m], \mathbb{W}) \rightarrow H^j(U, \mathbb{W})$$

induced by the inclusion $U \hookrightarrow \mathcal{C}_g^n[m]$ is an isomorphism when $j = 0, 1$ and an injection when $j = 2$.

Since the monodromy representation $\pi_1(\mathcal{C}_g^n[m], x_o) \rightarrow \mathrm{Sp}(H_{\mathbb{Q}})$ is Zariski dense, and since each irreducible representation of $\mathrm{Sp}(H_{\mathbb{Q}})$ is absolutely irreducible (so irreducible over \mathbb{Q}_p as well), the corollary holds when \mathbb{Q}_p is replaced by any field of characteristic zero. In particular, it holds when \mathbb{Q}_p is replaced by \mathbb{Q} .

7. PROOF OF THE STRICTNESS THEOREM

To prove Theorem 2.1, it suffices to prove the corresponding statement for M^c . Namely, that if D is an irreducible divisor in M^c and $h : E \rightarrow D$ is a dominant morphism from an irreducible smooth variety to D , then the image of $\pi_1(E, e_o)$ in Γ_o has finite index in the image of $\pi_1(M^c, x_o)$ in Γ_o . This suffices because $D \cap M$ is non-empty, as each component of $M^c - M$ has codimension ≥ 2 , and because $\pi_1(h^{-1}(D), e_o) \rightarrow \pi_1(E, e_o)$ is surjective as E is smooth.

7.1. Setup and Preliminaries. The assumption on the Picard group of M implies that $M = M^c = \mathbb{P}^1$ or $\dim M \geq 2$. Since \mathbb{P}^1 is simply connected, the theorem is trivially true when $\dim M = 1$. So we assume that $\dim M \geq 2$. Suppose that D is an irreducible divisor in M^c . Let X be a generic 2-dimensional linear section of \overline{M} (with respect to any projective imbedding) that is transverse to all strata of $(\overline{M} - M) \cup D$. Since each component of $\overline{M} - M^c$ has codimension ≥ 3 , X is a complete subvariety of M^c . Since each component of $M^c - M$ has codimension ≥ 2 , all singularities of X are isolated. The local Lefschetz Theorem [8, p. 153] implies that X is unibranch at each of its singular points.

Recall that $M_m \rightarrow M$ is the connected Galois covering that corresponds to the kernel of the mod- m monodromy homomorphism $\pi_1(M, x_o) \rightarrow \mathrm{Aut}(V_o/mV_o)$. Define X_m to be the covering of X that corresponds to the kernel of $\pi_1(X, x_o) \rightarrow \mathrm{Aut}(V_o/mV_o)$. It is a subvariety of M_m .

Define \overline{M}_m to be the normalization of \overline{M} in the function field of M_m . The morphism $\overline{M}_m \rightarrow \overline{M}$ is finite and surjective. Define M_m^c to be the inverse image of M^c in \overline{M}_m . The projection $M_m^c \rightarrow M^c$ is a Galois covering. It follows that M_m^c is also unibranch at each of its points.

Lemma 7.1. *The homomorphism $\pi_1(X_m, x_o) \rightarrow \pi_1(M_m^c, x_o)$ is surjective for all $m \geq 1$.*

Proof. Set $\overset{\circ}{X}_m = X_m \cap M_m$. Since X_m and M_m^c are unibranch at each of their points, Lemma 3.5 implies that the top and bottom morphisms in the diagram

$$\begin{array}{ccc} \pi_1(\overset{\circ}{X}_m, x_o) & \twoheadrightarrow & \pi_1(X_m, x_o) \\ \downarrow & & \downarrow \\ \pi_1(M_m, x_o) & \twoheadrightarrow & \pi_1(M_m^c, x_o) \end{array}$$

are surjective. The Theorem 3.1 implies that the left-hand vertical map is surjective. The result follows. \square

The fact that every component of $M_m^c - M_m$ has codimension ≥ 2 implies that every irreducible divisor in M^c is ample.

Lemma 7.2. *For all $m \geq 1$, every irreducible divisor in M_m^c is ample. Moreover, $\text{Pic } M_m^c$ is finitely generated and the restriction mapping $(\text{Pic } M_m^c) \otimes \mathbb{Q} \rightarrow (\text{Pic } M_m) \otimes \mathbb{Q}$ is an isomorphism.*

Proof. Fix an imbedding $M_m^c \hookrightarrow \mathbb{P}^r$. Suppose that W is an irreducible divisor in M_m^c . Since each component of $M_m^c - M_m$ has codimension ≥ 2 , $W' := W \cap M_m$ is non-empty. Corollary 3.4 implies that W' is ample in M_m . Since $\text{Pic } M_m$ has rank 1, there are positive integers d and m such that $\mathcal{O}_{M_m}(mW') \cong \mathcal{O}_{M_m}(d)$. A section of $\mathcal{O}_{M_m}(mW')$ is a rational section of $\mathcal{O}_{M_m^c}(d)$ over M_m^c that is regular on M_m . Suppose that $s \in H^0(M_m, \mathcal{O}_{M_m}(d))$ satisfying $\text{div}(s) = mW'$. Since all components of $M_m^c - M_m$ have codimension ≥ 2 , s must be regular on M_m^c and have divisor mW . So W is ample in M_m^c . The second assertion is easily proved and is left to the reader. \square

Corollary 7.3. *If W is an irreducible divisor in M_m^c such that $F = X_m \cap W$ is a complete curve, then $\pi_1(F, x_o) \rightarrow \pi_1(X_m, x_o)$ and $\pi_1(F, x_o) \rightarrow \pi_1(M_m^c, x_o)$ are surjective.*

Proof. Lemma 7.2 implies that W is an ample divisor in M^c . Let

$$X_F := X_m - \{x \in X_m^{\text{sing}} : x \notin F\}.$$

Then $F \subset X_F$. Since $X_F - F$ is smooth, it is a local complete intersection. The Lefschetz Theorem with Singularities [8, p. 153] implies that $\pi_1(F, x_o) \rightarrow \pi_1(X_F, x_o)$ is surjective. Since X_m is unibranch at each point of $X_m - X_F$, Lemma 3.5 implies that $\pi_1(X_F, x_o) \rightarrow \pi_1(X_m, x_o)$ is surjective. Consequently, $\pi_1(F, x_o) \rightarrow \pi_1(X_m, x_o)$ is also surjective. The final assertion follows from Lemma 7.1. \square

7.2. Proof of Theorem 2.1. Set $C = D \cap X$. Denote its normalization by $\pi : \tilde{C} \rightarrow C$. We may take x_o to be a smooth point of C . Denote its inverse image in \tilde{C} by \tilde{x}_o . Because C may have multiple branches at the singular points of X , the homomorphism $\pi_1(\tilde{C}, \tilde{x}_o) \rightarrow \pi_1(C, x_o)$ may not be surjective.

We will show that the image of $\pi_1(\tilde{C}, \tilde{x}_o)^\wedge \rightarrow \Gamma_o^\wedge$ has finite index in the image of the monodromy representation $\rho : \pi_1(C, x_o)^\wedge \rightarrow \Gamma_o^\wedge$, even though the image of $\pi_1(\tilde{C}, x_o)^\wedge$ in $\pi_1(C, x_o)^\wedge$ may have infinite index in $\pi_1(C, x_o)^\wedge$. This is the only point in the proof where we work with profinite fundamental groups. It is here where we use the Picard number 1 assumption.

Lemma 7.4. *Suppose that $m \geq 1$. If C' is a component of the inverse image C_m of C in X_m , then*

- (1) C' is an ample curve in X_m ,
- (2) $[C_m] = d[C'] \in (\text{Pic } X_m) \otimes \mathbb{Q}$, where d is the number of irreducible components of C_m ,
- (3) $\pi_1(C', x_o) \rightarrow \pi_1(M_m^c, x_o)$ is surjective.

Proof. Denote the inverse image of D in M_m by D_m . This may be reducible. Let W the irreducible component of D_m that contains C' . Since X is a generic linear

section of M_m^c , Bertini's Theorem [8, p. 151] implies that $X_m \cap W$ is irreducible and therefore equal to C' . Lemma 7.2 implies that W is ample in M_m , which implies that C' is ample in X_m .

Since $\text{Pic } M_m$ has rank 1, the classes $[D_m]$ and $[W]$ are proportional in $\text{Pic } M_m$ mod torsion. Their restrictions $[C_m]$ and $[C']$ to X_m are therefore proportional mod torsion. The second assertion follows as $X_m \rightarrow X$ is a Galois covering and the Galois group acts transitively on the components of C_m . The last assertion follows from Corollary 7.3. \square

For $m \geq 1$, set $\Gamma_{o,m} = \text{Aut}(V_o/mV_o)$. This is a finite group isomorphic to $\text{GL}_N(\mathbb{Z}/m\mathbb{Z})$.

The formulation of the following lemma and its proof were contributed by Madhav Nori. They avoid a gap in the proof of a previous version of the lemma.

Lemma 7.5. *For all $m \geq 1$, the index of the image of $\pi_1(\tilde{C}_o, \tilde{x}_o) \rightarrow \Gamma_{o,m}$ in the image of $\pi_1(C, x_o) \rightarrow \Gamma_{o,m}$ is bounded by $C \cdot C$.*

Proof. Set

$$G_m = \text{im}\{\pi_1(C, x_o) \rightarrow \Gamma_{o,m}\} \text{ and } K_m = \text{im}\{\pi_1(\tilde{C}_o, \tilde{x}_o) \rightarrow \Gamma_{o,m}\}.$$

Corollary 7.3 implies that G_m is also the image of $\pi_1(X_m, x_o)$ in $\Gamma_{o,m}$. Consequently, the inverse image C_m of C in X_m is a connected Galois covering of C . Choose a lift x'_o of x_o to C_m . Denote the component of C_m that contains x'_o by C' .

The group G_m acts on X_m with quotient X . The stabilizer of the component C' is K_m . Let $Y_m = K_m \backslash X_m$ so that we have the tower

$$X_m \longrightarrow Y_m \xrightarrow{h} X$$

of unramified coverings. Denote the image of C' in Y_m by C'' . Since K_m is the stabilizer of C' , the morphism $C'' \rightarrow C$ induced by h is birational. Lemma 7.4(2) implies that

$$h^*[C] = (\deg h)[C'] = [G_m : K_m][C'] \in (\text{Pic } Y_m) \otimes \mathbb{Q}$$

and that C'' is ample. Since $\deg h = [G_m : K_m]$, the projection formula implies that

$$[G_m : K_m](C'' \cdot C'') = (C'' \cdot h^*C) = (C \cdot C).$$

Since C'' is ample in Y_m , $C'' \cdot C'' > 0$, which gives the desired inequality

$$[G_m : K_m] = \frac{C \cdot C}{C'' \cdot C''} \leq C \cdot C.$$

\square

Combining this with Corollary 7.3, we obtain:

Corollary 7.6. *For all $m \geq 1$, the index of the image of $\pi_1(\tilde{C}, \tilde{x}_o)^\wedge \rightarrow \Gamma_o^\wedge$ in the image of $\pi_1(M^c, x_o)^\wedge \rightarrow \Gamma_o^\wedge$ is finite and bounded by $C \cdot C$.* \square

Denote the normalization of D by $\tilde{D} \rightarrow D$. Since X is a generic linear section of M^c and D , C is not contained in the singular locus of D . It follows that there is a smooth Zariski open subset U of C that lies in the smooth locus of D . It can

therefore be regarded as a curve in \tilde{D} . Corollary 7.6 and the commutativity of the diagram

$$\begin{array}{ccc} \pi_1(U, x_o) & \longrightarrow & \pi_1(\tilde{D}, x_o) \\ \downarrow & & \downarrow \\ \pi_1(\tilde{C}, \tilde{x}_o) & \longrightarrow & \pi_1(M^c, x_o) \end{array}$$

implies that $\text{im}\{\pi_1(\tilde{D}, x_o)^\wedge \rightarrow \Gamma_o^\wedge\}$ has finite index in $\text{im}\{\pi_1(M^c, x_o)^\wedge \rightarrow \Gamma_o^\wedge\}$. Theorem 2.1 now follows from Proposition 3.7.

8. COUNTEREXAMPLES

The following examples suggest that the hypotheses of Theorem 2.1 cannot easily be relaxed.

Example 8.1. Suppose that $g \geq 3$ and that $\ell \geq 3$. Suppose that $P \in \mathcal{A}_g[\ell]$. Let $M = M^c = \text{Bl}_P \mathcal{A}_g[\ell]$, the blow-up of $\mathcal{A}_g[\ell]$ at P . Take \mathbb{V} to be the standard local system over $\mathcal{A}_g[\ell]$. Take $E = D$ to be the exceptional divisor. Then the image of $\pi_1(E, e_o)$ in $\text{Sp}_g(\mathbb{Z})$ is trivial, so that the conclusions of Theorem 2.1 do not hold in this case. All hypotheses hold except for the condition (6) on Picard groups.

Example 8.2. One can also formulate a version of Theorem 2.1 for orbifolds. The genus 2 example presented in Section 5.4 with $\ell = 1$ shows that the conclusion of Theorem 2.1 fail in this case. Here $M = M^c = \mathcal{A}_2$. Condition (6), that $\text{rank Pic } M_m = 1$, holds when $m = 1$, but fails for at least some $m > 1$. The codimension of $\overline{M} - M^c$ is 2, so condition (4) also fails to hold.

The final example shows that if D is an irreducible smooth hyperplane section of a non-compact algebraic manifold M of dimension > 2 , then the inclusion $D \hookrightarrow M$ may not induce an isomorphism on fundamental groups. In such cases, Theorem 3.1 implies that D is not generic.

Example 8.3. Suppose that $\ell \geq 3$ is an odd integer. Since $\text{Sp}_3(\mathbb{Z})[\ell] \rightarrow \text{Sp}_3(\mathbb{Z}/2\mathbb{Z})$ is surjective, the hyperelliptic locus $\mathcal{H}_3[\ell]$ in $\mathcal{M}_3[\ell]$ is irreducible. Since the level ℓ subgroup of the mapping class group is torsion free, it follows that the level ℓ subgroup of the hyperelliptic mapping class group is also torsion free. This implies that $\mathcal{H}_3[\ell]$ is a smooth divisor in $\mathcal{M}_3[\ell]$. Since $\text{Pic } \mathcal{A}_3[\ell]$ has rank 1, the locus of hyperelliptic jacobians (the closure of the image of $\mathcal{H}_3[\ell]$ in $\mathcal{A}_3[\ell]$) is a hyperplane section of $\mathcal{A}_3[\ell]$. It follows that $\mathcal{H}_3[\ell]$ is the inverse image of a hyperplane section of $\mathcal{A}_3[\ell]$. The image of $\pi_1(\mathcal{H}_3[\ell], x_o) \rightarrow \pi_1(\mathcal{M}_3[\ell], x_o)$ and its profinite completion have infinite index as the first group is the level ℓ subgroup of the hyperelliptic mapping class group, which has infinite index in the genus 3 mapping class group. However, a result of A'Campo [1] implies that the image of the monodromy homomorphism $\pi_1(\mathcal{H}_3[\ell], x_o) \rightarrow \text{Sp}_3(\mathbb{Z})$ has finite index in $\text{Sp}_3(\mathbb{Z})$. The divisor $\mathcal{H}_3[\ell]$ is not generic as, for example, its image $\overline{\mathcal{H}_3}$ in \mathcal{M}_3 is not transverse to the singular locus of the boundary divisor Δ_0 of $\overline{\mathcal{M}_3}$.

9. DISCUSSION OF STRICTNESS THEOREMS

Consider the diagram

$$Z \xrightarrow{f} Y \xrightarrow{h} X$$

of morphisms of varieties where X and Z are smooth and f is dominant. Techniques developed by Deligne in [7, §8] to prove “strictness theorems” imply the following result:

Proposition 9.1. *The image of $H_1(Z; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z})$ has finite index in the image of $H_1(Y; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z})$.*

Since this result does not appear explicitly in the literature, we sketch a proof in the next section.

It would be very useful to know the extent to which this strictness result extends to (topological, profinite, proalgebraic, ...) fundamental groups. Choose base points x_o, y_o and z_o such that f and h are basepoint preserving. The most optimistic statement that one might hope to be true is that the image of $\pi_1(Z, z_o) \rightarrow \pi_1(X, x_o)$ has finite index in the image of $\pi_1(Y, y_o) \rightarrow \pi_1(X, x_o)$. If this were true, then Theorem 2.1 could be strengthened and its proof simplified. Unfortunately, it is not true, as we will show by example below. However, weaker statements that would follow from this optimistic statement, such as Theorem 2.1, do hold. We shall call them *non-abelian strictness theorems*. The only non-abelian strictness theorems of which I am aware are due to Nori [16], Lasell and Ramachandran [14], and Napier and Ramachandran [15]:

Theorem 9.2 (Nori (1983), Napier-Ramachandran (1998)). *Suppose that X and Y are connected, smooth projective varieties of positive dimension. If $Y \rightarrow X$ is a holomorphic immersion with ample normal bundle, then the image of $\pi_1(Y, y_o)$ in $\pi_1(X, x_o)$ has finite index.*

Theorem 9.3 (Lasell-Ramachandran (1996)). *If X, Y and Z are all irreducible and proper, then for each positive integer N there is a finite quotient Δ_N of $\pi_1(Y, y_o)$ such for all fields F and all reductive representations $\rho : \pi_1(X, x_o) \rightarrow \mathrm{GL}_N(F)$ such that*

$$\pi_1(Z, z_o) \xrightarrow{f_*} \pi_1(Y, y_o) \xrightarrow{h_*} \pi_1(X, x_o) \xrightarrow{\rho} \mathrm{GL}_N(F)$$

is trivial, the homomorphism $\rho \circ h_ : \pi_1(Y, y_o) \rightarrow \mathrm{GL}_N(F)$ factors through Δ_N :*

$$\begin{array}{ccccc} \pi_1(Z, z_o) & & & & \\ \downarrow f_* & \searrow \text{trivial} & & & \\ \pi_1(Y, y_o) & \xrightarrow{h_*} & \pi_1(X, x_o) & \xrightarrow{\rho} & \mathrm{GL}_N(F) \\ \downarrow & & & \nearrow & \\ \Delta_N & & & & \end{array}$$

The following example shows that several of the most optimistic non-abelian strictness statements given above are false.

9.1. The abelian strictness theorem does not extend naively to π_1 . We give a general method of constructing counterexamples to the most general forms of the non-abelian strictness assertion, and then use it to give an explicit counterexample. We will construct varieties

$$Z \xrightarrow{f} Y \xrightarrow{h} X$$

where X and Z are smooth and f is dominant, and a homomorphism $\rho : \pi_1(X, x_o) \rightarrow G(\mathbb{Q})$ with the property that the dimension of the Zariski closure of the image of $\pi_1(Z, z_o)$ in G has dimension strictly smaller than the dimension of the Zariski closure of the image of $\pi_1(Y, y_o)$ in G . This provides counter examples to the discrete, profinite and algebraic versions of the strictness assertion.

Suppose that G is a semi-simple \mathbb{Q} -group of adjoint type such that the symmetric space D of $G(\mathbb{R})$ is hermitian. Then D has a complex structure with the property that G acts on D by biholomorphisms. Suppose that Γ is an arithmetic subgroup of $G(\mathbb{Q})$. Denote by D° the open subset of D on which Γ acts fixed point freely. Suppose that D° is non-empty. Then $\Gamma \backslash D$ is a quasi-projective variety, and $X := \Gamma \backslash D^\circ$ is a smooth subvariety of it.

Suppose that H is a semi-simple \mathbb{Q} -subgroup of G , $H \neq G$, whose associated symmetric space is hermitian. Suppose that $d_o \in D$ is a point whose $H(\mathbb{R})$ -orbit D_H satisfies:

- (1) the image of D_H in $\Gamma \backslash X$ is a subvariety,
- (2) $d_o \in D_H^\circ := D_H \cap D^\circ$.

Set $Z = (\Gamma \cap H(\mathbb{Q})) \backslash D_H^\circ$. Then Z is a non-empty, smooth and the map $Z \rightarrow X$ induced by $D_H \rightarrow D$ is finite. Its image Y in X is closed. Denote the images of d_o in Z , Y and X by z_o , y_o and x_o , respectively.

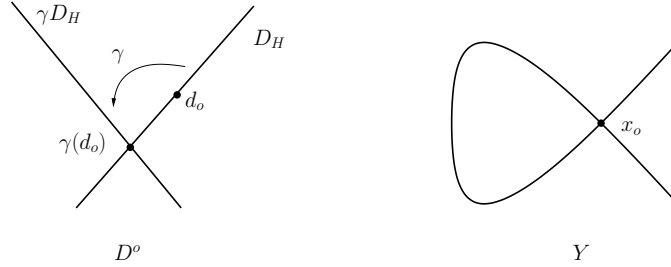


FIGURE 2. Y and part of its inverse image in D°

Note that there is an exact sequence

$$1 \rightarrow \pi_1(D^\circ, d_o) \rightarrow \pi_1(X, x_o) \rightarrow \Gamma \rightarrow 1.$$

There is therefore a natural homomorphism $\pi_1(X, x_o) \rightarrow G(\mathbb{Q})$.

Proposition 9.4. *If there exists $\gamma \in \Gamma$ such that $d_o \in D^\circ$ and*

- (1) $\gamma(d_o) \neq d_o$,
- (2) $\gamma(d_o) \in D_H$,
- (3) *no positive power γ^N of γ lies in $H(\mathbb{Q})$,*

then the dimension of the Zariski closure of the image of $\pi_1(Z, z_o)$ in G is larger than the dimension of the Zariski closure of the image of $\pi_1(Y, y_o)$ in G . Consequently, the image of $\pi_1(Z, z_o) \rightarrow \pi_1(X, x_o)$ has infinite index in the image of $\pi_1(Y, y_o) \rightarrow \pi_1(X, x_o)$.

Proof. The image of $\pi_1(Z, z_o)$ in $G(\mathbb{Q})$ is the arithmetic group $\Gamma_H := \Gamma \cap H(\mathbb{Q})$. Since Γ_H is an arithmetic subgroup of the semi-simple group H , its Zariski closure in G is H . Since $\gamma^N \notin H(\mathbb{Q})$ for all $N > 0$, $\gamma(D_H) \neq D_H$. Choose a path μ in D_H° from d_o to $\gamma(d_o)$. The image of μ in Y is a loop whose image under $\pi_1(Y, y_o) \rightarrow G(\mathbb{Q})$ is

γ . Since no positive power of γ lies in $H(\mathbb{Q})$, the cosets $\{\gamma^N H(\mathbb{Q}) : N \geq 1\}$ of H in G are distinct. It follows that the Zariski closure of the image of $\pi_1(Y, y_o) \rightarrow G(\mathbb{Q})$ has dimension strictly larger than the dimension of $H(\mathbb{Q})$. \square

Example 9.5. Let G be the \mathbb{Q} -group $\mathrm{PSL}_2 \times \mathrm{PSL}_2$, D be the product $\mathfrak{h} \times \mathfrak{h}$ of two copies of the upper half plane, and let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z})$. Let

$$g_1 = \begin{pmatrix} 0 & -1/2 \\ 1 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

These are elements of $\mathrm{GL}_2(\mathbb{Q})$ with positive determinant and therefore act on \mathfrak{h} . Set $d_o = (g_1(i), g_2(i))$. Since $-1/g_1(i) = 2 + 2i$ and $-1/g_2(i) = 3i$, the $\mathrm{SL}_2(\mathbb{Z})$ orbits of $g_1(i)$ and $g_2(i)$ both intersect $\mathrm{Im} \tau > 1$, which implies that their isotropy groups in $\mathrm{PSL}_2(\mathbb{Z})$ are trivial. Thus $d_o \in D^\circ$.

Set $g = (g_1, g_2)$. Denote the diagonal copy of PSL_2 in $\mathrm{PSL}_2 \times \mathrm{PSL}_2$ by Δ_G . Let $H = g\Delta_G g^{-1}$ and D_H be the H -orbit of d_o . Define $\gamma = (\gamma_1, \gamma_2)$, where

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Since $\mathrm{trace}(\gamma_1) = 2$ and $\mathrm{trace}(\gamma_2) = 4$, both act fixed point freely on \mathfrak{h} . In particular, $\gamma(d_o) \neq d_o$. If γ^N is conjugate to an element of H , then there exists $h \in \mathrm{PSL}_2(\mathbb{Q})$ such that $\gamma_1^N = g_1 h^N g_1^{-1}$ and $\gamma_2^N = g_2 h^N g_2^{-1}$. In particular, γ_1^N is conjugate to γ_2^N in $\mathrm{PSL}_2(\mathbb{Q})$. Since γ_1 has eigenvalue 1, and γ_2 has eigenvalues $2 \pm \sqrt{3}$, γ_1^N is conjugate to γ_2^N if and only if $N = 0$.

9.2. Proof of the Abelian Strictness Assertion. All cohomology in this section will be with rational coefficients. Recall from [7] that the rational cohomology of a (complex algebraic) variety T has a natural weight filtration

$$0 = W_{-1}H^j(T) \subseteq \cdots \subseteq W_r H^j(T) \subseteq W_{r+1}H^j(T) \subseteq \cdots \subseteq W_{2j}H^j(T) = H^j(T)$$

Its r th graded quotient $W_r H^j(T)/W_{r-1}$ will be denoted by $\mathrm{Gr}_r^W H^j(T)$. It has the property that $W_{j-1}H^j(T) = 0$ when T is smooth, and $H^j(T) = W_j H^j(T)$ when T is proper.

If $f : S \rightarrow T$ is a morphism of complex algebraic varieties, then the induced morphism

$$f^* : H^\bullet(T) \rightarrow H^\bullet(S)$$

is *strict* with respect to the weight filtration W_\bullet . That is,

$$\mathrm{im}\{f^* : H^j(T) \rightarrow H^j(S)\} \cap W_r H^j(T) = \mathrm{im}\{f^* : W_r H^j(T) \rightarrow H^j(S)\}.$$

Proposition 9.6. *If $f : Z \rightarrow Y$ is a dominant morphism from a smooth variety, then*

$$0 \longrightarrow W_0 H^1(Y) \longrightarrow H^1(Y) \xrightarrow{f^*} H^1(Z)$$

is exact.

Proposition 9.1 is an immediate consequence. Since $H_1(X; \mathbb{Z})$, $H_1(Y; \mathbb{Z})$ and $H_1(Z; \mathbb{Z})$ are finitely generated abelian groups, it suffices to prove that the images of $H_1(Z; \mathbb{Q})$ and $H_1(Y; \mathbb{Q})$ in $H_1(X; \mathbb{Q})$ are equal. We will prove the dual assertion; namely:

$$\ker\{h^* : H^1(X) \rightarrow H^1(Y)\} = \ker\{f^* h^* : H^1(X) \rightarrow H^1(Z)\}.$$

To prove this, consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_0 H^1(Y) & \longrightarrow & H^1(Y) & \xrightarrow{f^*} & H^1(Z) \\ & & & & \uparrow h^* & \nearrow & \\ & & & & H^1(X) & & \end{array}$$

Since X is smooth, $W_0 H^1(X) = 0$. Strictness and the exactness of the top row of the diagram imply that

$$\operatorname{im} h^* \cap \ker f^* = \operatorname{im} h^* \cap W_0 H^1(Y) = \operatorname{im} W_0 H^1(Z) = 0.$$

This implies that $\operatorname{im} f^* = \operatorname{im}(f^* h^*)$ as required.

Sketch of Proof of Proposition 9.6. This proof requires familiarity with Deligne's construction [7] of the mixed Hodge structure (MHS) on the cohomology of a general complex algebraic variety. The first step is to observe that we may assume that $Z \rightarrow Y$ is proper and surjective. If not, one can find a smooth variety Z' that contains Z as a Zariski open subset and a proper surjective morphism $h' : Z' \rightarrow Y$ that extends h . Since Z' is smooth, standard arguments imply that $H^1(Z') \rightarrow H^1(Z)$ is injective.

Suppose that $h : Z \rightarrow Y$ is surjective. Fix a completion \overline{Y} of Y . Then there exists a smooth completion \overline{Z} of Z such that $\overline{Z} - Z$ is a normal crossings divisor D in \overline{Z} , and a morphism $\overline{h} : \overline{Z} \rightarrow \overline{Y}$ that extends h .

Set $Y_0 = \overline{Y} - Y$. Then $(\overline{Z}, D) \rightarrow (\overline{Y}, Y_0)$ can be completed to a hypercovering $(\overline{M}_\bullet, D_\bullet)$ of (\overline{Y}, Y_0) where

- (1) each \overline{M}_n is smooth and proper,
- (2) D_n is a normal crossings divisor in \overline{M}_n ,
- (3) $(M_0, D_0) = (\overline{Z}, D)$.

Set $M_n = \overline{M}_n - D_n$. The spectral sequence of the associated simplicial variety satisfies

$$E_1^{s,t} = H^t(M_s) \implies H^{s+t}(Y).$$

Since it is a spectral sequence in the category of MHS

$$H^1(Y)/W_0 = \ker\{H^1(M_0) \xrightarrow{d_1} H^1(M_1)\}.$$

The result follows as $Z = M_0$. □

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